Print your name: ____________________

Score

1. Use either cylindrical or spherical coordinates to find the mass of the region that is above the plane $z = 0$, outside of the cone $x^2 + y^2 = z^2$, and inside the sphere $x^2 + y^2 + z^2 = 2$ and has density function z.

Solution: The answer is $\pi/2$.

Spherical coordinates: The equation of the sphere is $\rho = \sqrt{2}$ and the equation of the cone is $\cos\phi = \sin\phi$, i.e., $\phi = \pi/4$. Since $z = \rho\cos\phi$, and the Jacobian is $\rho^2 \sin\phi$, we have the integral

$$\int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\sqrt{2}} \rho^3 \sin\phi \cos\phi \, d\rho \, d\phi \, d\theta.$$

Cylindrical coordinates: You may be tempted to put the dz on the inside, but that won't work unless you split the integral into two integrals, because for some points in the xy plane the region is bounded above by the cone, and for other points the region is bounded above by the sphere. You can do it all in one by putting dr on the inside and noting that r ranges from the cone ($r = z$) to the sphere ($r^2 + z^2 = 2$) at each value of z and θ. So we have

$$\int_0^{2\pi} \int_0^1 \int_z^{\sqrt{2-z^2}} zr \, dr \, dz \, d\theta.$$

BONUS (worth 500 feel-special points, 0 class points): Write the same integral in spherical coordinates if you used cylindrical above, or vice versa.