Laplace transform with a Heaviside function

by Nathan Grigg

The formula

To compute the Laplace transform of a Heaviside function times any other function, use

$$\boxed{\mathcal{L}\Big\{u_c(t)f(t)\Big\} = e^{-cs}\mathcal{L}\Big\{f(t+c)\Big\}.}$$

Think of it as a formula to get rid of the Heaviside function so that you can just compute the Laplace transform of $f(t+c)$, which is doable.

In words: To compute the Laplace transform of u_c times f, shift f left by c, take the Laplace transform, and multiply the result by e^{-cs}. Remember that to shift left, you replace t with $t+c$.

The other way to write the formula

You will sometimes see the formula written as $\mathcal{L}\{u_c(t)f(t-c)\} = e^{-cs}F(s)$, where $F(s)$ is the Laplace transform of $f(t)$. This is a correct formula that says the same thing as the first formula, but it is a *terrible* way to compute the Laplace transform. It is, however, a perfectly fine way to compute the *inverse* Laplace transform. Rewrite it as

$$\boxed{\mathcal{L}^{-1}\Big\{e^{-cs}F(s)\Big\} = u_c(t)f(t-c).}$$

In words: To compute the inverse Laplace transform of e^{-cs} times F, find the inverse Laplace transform of F, call it f, then shift f right by c and multiply by u_c. Remember that to shift right, you replace t with $t-c$.

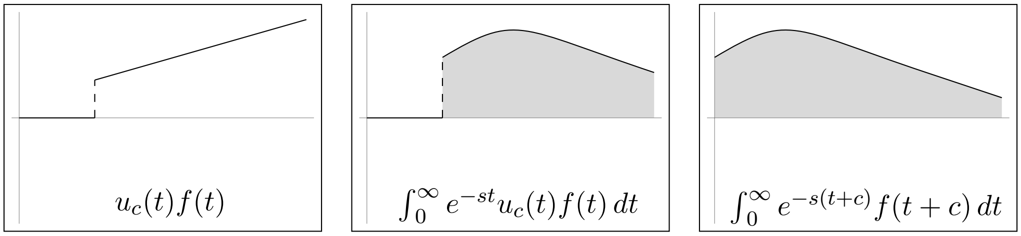
Why it works

Right now you are probably thinking, *Don't prove it to me! I trust you!* Mathematicians believe that understanding a proof is crucial to understanding a statement, because that's how our brains work. Sometimes we go a little too far and forget that there are other great ways to understand math.

In this case, though, you should see the proof. It will help you understand. *Trust me!*

$$\begin{aligned}
\mathcal{L}\Big\{u_c(t)f(t)\Big\} &= \int_0^\infty e^{-st}u_c(t)f(t)\,dt && \text{definition of Laplace transform} \\
&= \int_c^\infty e^{-st}f(t)\,dt && \text{because the integral from 0 to } c \text{ is } 0 \\
&= \int_0^\infty e^{-s(t+c)}f(t+c)\,dt && \text{shift left by } c \\
&= e^{-cs}\int_0^\infty e^{-st}f(t+c)\,dt && \text{pull out constant } e^{-cs} \\
&= e^{-cs}\mathcal{L}\Big\{f(t+c)\Big\} && \text{definition of Laplace transform}
\end{aligned}$$

Here is the same thing in pictures:



The last integral simplifies to $e^{-cs}\mathcal{L}\{f(t+c)\}$ because at this point we are treating s as a constant.

Hopefully by looking at these pictures, you see the key idea: *Shifting left by c allows us to get rid of the Heaviside function.*

Some algebra required

If $f(t) = t^2$, then $f(t+c) = (t+c)^2$. How do you take the Laplace transform of $(t+c)^2$? You have to rearrange things, in this case expanding to $t^2 + 2ct + c^2$.

Another example: If $f(t) = e^{2t}$ then $f(t+c) = e^{2(t+c)}$. If you want to take the Laplace transform, you need to expand to $e^{2t}e^{2c}$.

And a harder one: If $f(t) = \sin t$, then $f(t+c) = \sin(t+c)$. If you want to take the Laplace transform, you need to do some trigonometric magic. If c is a multiple of $\pi/2$ or π, you can probably figure it out by drawing some triangles. Otherwise, pull out your trig identities![1]

This is not a product rule

One common misconception about this Laplace transform formula is that it is a kind of product rule, that the Laplace transform of $u_c(t)$ times $f(t)$ is the Laplace transform of $u_c(t)$ times the Laplace transform of $f(t)$. *It is not! There is no product rule for Laplace transforms.*

Still not convinced? Here you go:

$$\mathcal{L}\{u_c(t)\} = \frac{e^{-cs}}{s} \qquad \mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{u_c(t)\cdot t\} = e^{-cs}\left(\frac{1}{s^2} + \frac{c}{s}\right)$$
$$\mathcal{L}\{t\} = \frac{1}{s^2} \qquad \mathcal{L}\{t^2\} = \frac{2}{s^3} \qquad \mathcal{L}\{t\cdot t^2\} = \frac{6}{s^4}$$

Once again, there is no product rule. There is only a fancy way to take a step function from inside a Laplace transform operator and bring it outside.

[1] You might need $\sin(t+c) = \cos(c)\sin(t) + \sin(c)\cos(t)$, or maybe $\cos(t+c) = \cos(c)\cos(t) - \sin(c)\sin(t)$.