Laplace Transform Practice

Solve the following differential equation

$$y'' + 4y = \begin{cases} \sin t & t \le 9\pi \\ 0 & t > 9\pi \end{cases}, \quad y(0) = 0, \quad y'(0) = -1$$

Determine the amplitude of the oscillation when $t > 9\pi$.

Answer The differential equation can be written

$$y'' + 4y = \sin t - u_{9\pi}(t) \sin t$$

Applying the Laplace transform, we get

$$s^2 Y + 1 - 4Y = \frac{1}{s^2+1} + \frac{1}{s^2+1} e^{-9\pi s}$$

Note the plus instead of minus between the two fractions, which comes because the final Laplace transform is computed as

$$\mathcal{L}\{u_{9\pi}(t)\sin t\} = e^{-9\pi s}\mathcal{L}\{\sin(t+9\pi)\} = e^{-9\pi s}\mathcal{L}\{-\sin t\} = -e^{-9\pi s}\frac{1}{s^2+1}$$

Solving for Y, we get

$$\frac{1}{(s^2+1)(s^2+4)}(1+e^{-9\pi s}) - \frac{1}{s^2+4} = \left(\frac{1/3}{s^2+1} - \frac{1/3}{s^2+4}\right)(1+e^{-9\pi s}) - \frac{1}{s^2+4}$$

Taking inverse Laplace transforms, we get

$$\begin{aligned} &\frac{1}{3}\sin t - \frac{1}{6}\sin 2t + \left(\frac{1}{3}\sin(t-9\pi) - \frac{1}{6}\sin(2(t-9\pi))\right)u_{9\pi}(t) - \frac{1}{2}\sin 2t \\ &\qquad = \frac{1}{3}\sin t - \frac{2}{3}\sin 2t + \left(-\frac{1}{3}\sin t - \frac{1}{6}\sin 2t\right)u_{9\pi}(t) \end{aligned}$$

If $t > 9\pi$ then $u_{9\pi}(t) \equiv 1$, and the equation is $-(5/6)\sin 2t$, which has amplitude $5/6$.

(over)

Consider the initial value problem

$$y'' + y = \sum_{k=1}^{20} \delta_{k\pi}(t), \quad y(0) = 0, \quad y'(0) = 0.$$

Without solving the differential equation, try to describe the oscillation. What will happen when $t > 20\pi$? (Hint: try to graph the solution from 0 to 4π.)

Now solve the differential equation using Laplace transforms.

Answer We take the Laplace transform of both sides to get

$$(s^2 + 1)Y = \sum_{k=1}^{20} e^{-k\pi s}$$

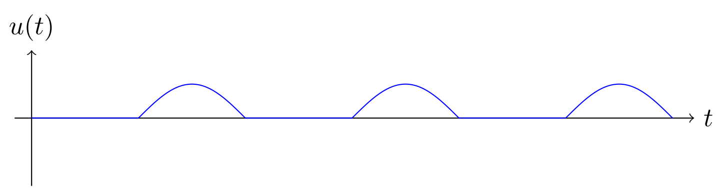
Solving for Y, we get

$$Y = \sum_{k=1}^{20} \frac{e^{-k\pi s}}{s^2 + 1}$$

We take the inverse Laplace transform to get

$$y(t) = \sum_{k=1}^{20} \sin(t - k\pi)u_{k\pi}(t) = \sin(t - \pi)u_{\pi}(t) + \sin(t - 2\pi)u_{2\pi}(t) + \sin(t - 3\pi)u_{3\pi}(t) + \cdots$$

The graph looks like this:



Consider the initial value problem

$$y'' + y = \sum_{k=1}^{20} \delta_{k\pi/2}(t), \quad y(0) = 0, \quad y'(0) = 0.$$

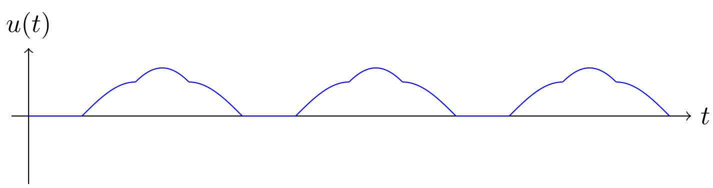
As before, try to describe the oscillation. What will happen when $t > 20\pi$?

Now solve the differential equation using Laplace transforms.

Answer This is pretty much the same as the last problem, except we get

$$y(t) = \sum_{k=1}^{20} \sin(t - k\pi/2) u_{k\pi/2}(t)$$

The graph looks like:



Notice the points every $\pi/2$ where the system gains (or loses) a jolt of energy.