Newton's Law of Cooling

If you have an insulated box with internal temperature $u(t)$ and external temperature T, then Newton's law of cooling says that the change in internal temperature is proportional to the difference between the inside and outside temperatures. It assumes that T is not affected by this heat transfer.

Write a differential equation that describes $u(t)$ in terms of T and a positive constant k.

Solution. The differential equation is $u' = k(T - u)$ because when the outside temperature T is greater than the inside temperature u, then the inside temperature should increase, meaning u' should be positive.

Suppose that $u(0) = 50°\text{C}$ and $T = 10°\text{C}$. Solve the initial value problem and determine how long it will take until $u(t) = 11°\text{C}$. Your answer will depend on k.

Solution. With T a constant, this is both separable and linear. Whichever way you solve it, the general solution is $10 + ce^{-kt}$. The constant is 40, so the solution to the initial value problem is $10 + 40e^{-kt}$. This is equal to 11 when $t = \ln(40)/k$.

Now suppose that T is not constant, but depends on time. Solve the initial problem again with the same initial conditions, but with $T = 10 + 10 \sin t$. To make things simpler, let $k = 1$.

Solution. The differential equation is now $u' + u = 10 + 10 \sin t$, which is linear. The integrating factor is e^t, which gives us $e^t u = 10 \int (e^t + e^t \sin t)\, dt$. We can integrate $e^t \sin t$ either using integration by parts twice, or using the formula $\int e^{at} \sin bt\, dt = e^{at}(a \sin bt - b \cos bt)/(a^2 + b^2)$ to get

$$\begin{aligned} e^t u &= e^t(10 + 5 \sin t - 5 \cos t) + c \\ u &= \underbrace{10 + 5 \sin t - 5 \cos t}_{\text{steady state}} + \underbrace{ce^{-t}}_{\text{transient}} \end{aligned}$$

Using the initial condition we get $c = 45$.

Separate your solution into two parts: the *transient* part, which become 0 as t gets large, and the *steady state* solution, which will oscillate forever. Compare the steady state solution to the function $T = 10 + 10 \sin t$. How are they similar? How are they different?

Solution. Using trig identities, you can write $5 \sin t - 5 \cos t = 5\sqrt{2} \sin(t - \pi/2)$. So the steady state solution has the same period as T, but has smaller amplitude (7.1 compared to 10) and is phase is shifted by $\pi/2$.