Math 307 Worksheet: 3.7-3.8 Solutions

1. The differential equation is $\frac{1}{4}u''+\frac{1}{2}u'+\frac{5}{4}u=0$. This has a general solution of $e^{-t}(c_1\cos(2t)+c_2\sin(2t))$. Using the initial conditions $u(0)=1$ and $u'(0)=-3$, we get $c_1=1$ and $c_2=-1$, so the solution is
$$u=e^{-t}(\cos(2t)-\sin(2t)).$$
If we want, we can rewrite $\cos(2t)-\sin(2t)$ as $\sqrt{2}\cos(2t+\pi/4)$.

2. We need $u=0$, since e^-t is never zero, this means we need $\cos(2t)-\sin(2t)=0$, or in other words $\cos(2t)=\sin(2t)$. This happens when $\tan(2t)=1$, which is when $2t=\pi/4$, so $t=\pi/8$.
We could also set $\cos(2t+\pi/4)=0$. Cosine is zero at $\pi/2$, so we get $2t+\pi/4=\pi/2$, or again $t=\pi/8$.
The period of oscillation is $(2\pi)/2=\pi$, and an object passes through the equilibrium point every half-period. So the mass will cross the equilibrium for the 3rd time 1 period after it crosses the first time: when $t=\pi/8+\pi=9\pi/8$.

3. The mass oscillates back and forth between $\sqrt{2}e^{-t}$ and $-\sqrt{2}e^{-t}$, so as long as $\sqrt{2}e^{-t}<1/12$, then the object stays within 1 inch of equilibrium. Since $\sqrt{2}e^{-t}$ is a decreasing function, we can just solve for the time that $\sqrt{2}e^{-t}=1/12$, which is when $t=-\ln\left(\frac{1}{12\sqrt{2}}\right)=\ln(12\sqrt{2})\approx 2.83$.

4. The particular solution will be of the form $Y=A\sin t+B\cos t$. Plugging in to the differential equation we get equations
$$A-\frac{1}{2}B=\frac{1}{2}$$
$$\frac{1}{2}A+B=0$$
So $A=2/5$ and $B=-1/5$. This gives us a general solution of
$$e^{-t}(c_1\cos(2t)+c_2\sin(2t))+(2/5)\sin t-(1/5)\cos t$$
Using our initial conditions, we get a system
$$c_1-\frac{1}{5}=1$$
$$-c_1+2c_2+\frac{2}{5}=-3$$
Solving, we get $c_1=6/5$ and $c_2=-11/10$.
The steady state solution is $(2/5)\sin t-(1/5)\cos t$ which has frequency equal to that of the forcing function and amplitude of approximately 0.45, which is slightly less than the amplitude of the forcing function.