ANSWER TO EIGENBASIS EXERCISES

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(1) The eigenvalues of A are $3, 3, 0$. A basis for E_3 is $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ and a basis for E_0 is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$, so an eigenbasis is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

We can write $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in terms of the eigenbasis as

$$- \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

So we have

$$A^{10}\mathbf{x} = -(3^{10}) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 2(3^{10}) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3(0^{10}) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 3^{10} \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.$$

(2) The eigenvalues of A are $-1, 0, 1$ with corresponding eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

This is our eigenbasis. Now we write

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = (x - z) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-x + y + z) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (x - y) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

So we have

$$A^t\mathbf{x} = (-1)^t(x - z) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 1^t(x - y) \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-1)^t(x - z) + x - y \\ (-1)^t(x - z) \\ x - y \end{bmatrix}$$