EIGENBASES

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Let A be an $n \times n$ matrix. An **eigenbasis** corresponding to A is a basis for R^n consisting entirely of eigenvectors for A. Such a basis only exists if A is diagonalizable (or not defective). To find an eigenbasis, you find a basis for each eigenspace of A. The collection of all these basis vectors for an eigenbasis for A.

The fact that you get a basis for R^n is kind of special. Usually, combining two linearly independent sets doesn't give you a new linearly independent set, but it does when each linearly independent set consists of eigenvectors, each set for a different eigenvalue. You can prove this in pretty much the same way you would prove that any two eigenvectors that have different eigenvalues are linearly independent.

Example. The matrix $A = \begin{bmatrix} -83 & 4 & 35 \\ -348 & 20 & 140 \\ -174 & 8 & 74 \end{bmatrix}$ has eigenvalues $3, 4, 4$. The eigenspace E_4 has a basis $\left\{ \begin{bmatrix} 35 \\ 0 \\ 87 \end{bmatrix}, \begin{bmatrix} 4 \\ 87 \\ 0 \end{bmatrix} \right\}$, and the eigenspace E_3 has a basis $\left\{ \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$. So an eigenbasis corresponding to A is $\left\{ \begin{bmatrix} 35 \\ 0 \\ 87 \end{bmatrix}, \begin{bmatrix} 4 \\ 87 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right\}$.

Exercises.

(1) Find an eigenbasis corresponding to $A = \begin{bmatrix} 3 & 0 & -3 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}$. Write $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ as a linear combination of the basis and use this to find $A^{10}\mathbf{x}$.

(2) Find an eigenbasis corresponding to $A = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. Write $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ as a linear combination of the basis and use this to find a general formula for $A^t\mathbf{x}$.