1. (5 points) For exactly $1 at Trader Joe's, you can buy either (a) 1 apple, 1 banana, and 1 orange, (b) 2 bananas and 3 oranges, or (c) 1 apple and 2 oranges. How much for just one apple?

$$\begin{aligned} x + y + z &= 1 \\ 2y + 3z &= 1 \\ x \qquad + 2z &= 1 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 1 \\ 1 & 0 & 2 & 1 \end{array}\right]$$

$$\Big\downarrow \text{Row Reduce}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3/5 \\ 0 & 1 & 0 & 1/5 \\ 0 & 0 & 1 & 1/5 \end{array}\right]$$

$x = 3/5$, so an apple costs $0.60

2. (5 points) For what value(s) of a is the following system of equations consistent?

$$\begin{aligned} x + 2y &= 1 \\ 2x + 4y &= a \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 4 & a \end{array}\right] \xrightarrow[\text{Reduce}]{\text{Row}} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & a-2 \end{array}\right]$$

For the system to be consistent, you need $a=2$

3. (5 points) Suppose $A^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $AB + AC = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}$.

If possible, find $B + C$.

$$A^{-1}(AB+AC) = B+C.$$

$$\text{So } B+C = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 5 & 4 \\ 2 & 1 \end{bmatrix}$$

4. (5 points) Calculate the inverse of $D = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$.

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row Reduce}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 2 \end{array}\right]$$

5. (6 points) Consider the following systems of equations:

I. A 4×5 homogeneous system. ∞

II. A 4×2 consistent system. 1 or ∞

III. A 3×4 system. 0 or ∞

(An $m \times n$ system of equations has m equations and n variables.)

Answer the following:

(a) Which of these WILL ALWAYS have at least one solution?

a) I only b) II only c) III only d) I and II only
e) I and III only f) II and III only g) I, II, and III h) None of these

(b) Which of these MAY have infinitely many solutions?

a) I only b) II only c) III only d) I and II only
e) I and III only f) II and III only g) I, II, and III h) None of these

(c) Which of these WILL ALWAYS have infinitely many solutions?

a) I only b) II only c) III only d) I and II only
e) I and III only f) II and III only g) I, II, and III h) None of these

6. (4 points) Answer each of the following

(a) Suppose A, B, and C are nonsingular matrices, and $C^{-1} = AB$. Which one of the following is equal to A^{-1}?

a) CB b) $B^{-1}C$ c) BC d) $C^{-1}B$

$$A^{-1}C^{-1} = B \Rightarrow A^{-1} = BC$$

(b) True or false: If $AC = BC$, then $A = B$.

False: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$

7. (5 points) Is the set of vectors $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$ linearly dependent or independent?

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} \xrightarrow[\text{Reduce}]{\text{Row}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So it is <u>linearly</u> <u>independent</u>

8. (5 points) Let $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ be a set of four vectors in $\mathbf{R}^3$. Prove that the set is linearly dependent. (Do not use the theorem that states a set of more than n vectors in $\mathbf{R}^n$ is linearly dependent.)

$a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 + a_4\vec{v}_4 = \vec{0}$ is a system of equations with four variables and three equations so it must have more than one solution.

9. (5 points) Let N be a square matrix with $N^2 = \mathcal{O}$. Prove that N is singular. (Remember that N^2 means N times N, and $\mathcal{O}$ is the zero matrix.)

Let $\vec{x}$ be a nonzero vector.

Since $N^2 = \mathcal{O}$, $N^2\vec{x} = \vec{0}$

If $N\vec{x} = \vec{0}$ then N is singular.

If not, then $N(N\vec{x}) = \vec{0}$ which means that N is singular.

Nonzero vector.