1. (10 points) Let $A = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 1 & 1 & -1 & -2 \\ 3 & 1 & 3 & 2 \\ 2 & 2 & -2 & -4 \end{bmatrix}$. $\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & -3 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(a) Find a basis for the row space of $A$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ -4 \end{bmatrix} \right\}$$

This is one possible answer.
There are many others.

(b) Find a basis for the null space of $A$.

$$x_1 = -2x_3 - 2x_4$$
$$x_2 = 3x_3 + 4x_4$$
$$x_3 = x_3$$
$$x_4 = x_4$$

$$\vec{x} = \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} -2 \\ 4 \\ 0 \\ 1 \end{bmatrix}$$

these two vectors give a basis

(c) What is the rank of $A$?

2

(d) What is the nullity of $A$?

2

2. (5 points) Find an orthogonal basis for the 3-dimensional subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, $\begin{bmatrix}3\\0\\1\\-1\end{bmatrix}$, and $\begin{bmatrix}5\\1\\-3\\-1\end{bmatrix}$.

$$u_1 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$$

$$u_2 = \begin{bmatrix}3\\0\\1\\-1\end{bmatrix} - \frac{\overset{2}{\langle 3,0,1,-1\rangle \cdot \langle 1,0,0,1\rangle}}{\underset{2}{\langle 1,0,0,1\rangle \cdot \langle 1,0,0,1\rangle}} \begin{bmatrix}1\\0\\0\\1\end{bmatrix} = \begin{bmatrix}2\\0\\1\\-2\end{bmatrix}$$

$$u_3 = \begin{bmatrix}5\\1\\-3\\-1\end{bmatrix} - \frac{\overset{4}{\langle 5,1,-3,-1\rangle \cdot \langle 1,0,0,1\rangle}}{\underset{2}{\langle 1,0,0,1\rangle \cdot \langle 1,0,0,1\rangle}} \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \frac{\overset{9}{\langle 5,1,-3,-1\rangle \cdot \langle 2,0,1,-2\rangle}}{\underset{9}{\langle 2,0,1,-2\rangle \cdot \langle 2,0,1,-2\rangle}} \begin{bmatrix}2\\0\\1\\-2\end{bmatrix}$$

$$= \begin{bmatrix}5-2-2\\1-0-0\\-3-0-1\\-1-2+2\end{bmatrix} = \begin{bmatrix}1\\1\\-4\\-1\end{bmatrix}.$$

3. (9 points) For each matrix described below, determine the dimension of the null space and the dimension of the range.

(a) $A = \begin{bmatrix} 1 & -1 & 3 & 5 & -2 & 10 & -2 \\ 0 & 1 & -3 & 0 & 1 & 9 & 3 \end{bmatrix}$.

Dimension of null space of $A$: 5   Dimension of range of $A$: 2

(b) $B$ is the $3 \times 3$ matrix such that $B\mathbf{x}$ is the reflection of $\mathbf{x}$ across the plane $x + y = 0$.

It is not necessary to find exactly what B is to know these dimensions.

Dimension of null space of $B$: 0   Dimension of range of $B$: 3

(c) $C$ is the $2 \times 3$ matrix such that $C \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ z \end{bmatrix}$.

Dimension of null space of $C$: 1   Dimension of range of $C$: 2

4. (6 points) Let $C$ be the matrix from question 3(c).

(a) Find a basis for the null space of $C$.

$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Find a basis for the range of $C$.

Any basis for $\mathbb{R}^2$, e.g. $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

5. (5 points) Use the least squares method to find a linear function that best approximates the following data:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $y$ | $2$ | $1$ | $-1$ | $0$ | $-1$ |

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \qquad A^T\vec{b} = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$$

Solve $A^TA\vec{x}^* = A^T\vec{b}$:

$$\left[\begin{array}{cc|c} 5 & 0 & 1 \\ 0 & 10 & -7 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 1/5 \\ 0 & 1 & -7/10 \end{array}\right]$$

So $\vec{x}^* = \begin{bmatrix} 1/5 \\ -7/10 \end{bmatrix}$

The eq. of the line is $\boxed{y = -\frac{7}{10}x + \frac{1}{5}}$.

6. (5 points) Let $W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y + z = 0 \right\}$.

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two nonzero vectors in $W$ with $\mathbf{v}_1^T \mathbf{v}_2 = 0$. Prove that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $W$.

If $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{0}$, then dotting each side with $\vec{v}_1$, we get $a_1 \vec{v}_1^T \vec{v}_1 = \vec{0}$. So $a_1 = 0$. Similarly, $a_2 = 0$. So $\{\vec{v}_1, \vec{v}_2\}$ is a linearly independent set.

W is 2-dimensional, and any set of 2 linearly independent vectors in a 2-dimensial space form a basis.

7. (5 points) Let $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0 \text{ and } y \geq 0 \right\}$.

For each of the following, answer yes or no. No explanation is necessary.

(a) Is the zero vector $\mathbf{0}$ in $S$?

Yes

(b) If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in $S$, is $\mathbf{v}_1 + \mathbf{v}_2$ in $S$?

Yes

(c) If $\mathbf{v}$ is in $S$ and $a$ is a real number, is $a\mathbf{v}$ in $S$?

No. $a = -1$ and $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow a\vec{v} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin S$