1. (a) $A^{-1} = \begin{bmatrix} 1 & 0 & -1/b \\ 0 & 1/a & 0 \\ 0 & 0 & 1/b \end{bmatrix}$

   (b) Yes, because matrix whose columns are these three vectors has determinant of 4, so the matrix is nonsingular, so the vectors are linearly independent.

   (c) One possible answer: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

2. (a) The characteristic polynomial is $\begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2(1-\lambda)$, so the eigenvalues are 2 and 1.

   E_2 is the nullspace of $\begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, so a basis for E_2 is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$.

   E_1 is the nullspace of $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$, so a basis for E_2 is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$.

   (b) Yes, because 2 has an algebraic multiplicity of 2 but a geometric multiplicity of 1.

3. (a) If we expand along the first column, we get

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1 - 2 = \boxed{-1}$$

   (b) $\det(AB) = \det A \det B = (-1)(5) = \boxed{-5}$

   (c) 1/5

   (d) True, since $\det A \neq 0$.

4. (a) We have $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}$, $\vec{\mathbf{b}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $A^TA = \begin{bmatrix} 21 & 27 \\ 27 & 35 \end{bmatrix}$, $A^T\vec{\mathbf{b}} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$. Solving the system of equations, we get $\vec{\mathbf{x}}^* = \begin{bmatrix} 5/6 \\ -1/2 \end{bmatrix}$.

   (b) $A\vec{\mathbf{x}}^* - \vec{\mathbf{b}} = \begin{bmatrix} 1/3 \\ 1/6 \\ 5/6 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/6 \\ -1/6 \end{bmatrix}$, which has length $\sqrt{1/6} \approx .4082$.

5. A has eigenvalues 2 and -1 with corresponding eigenvectors $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, respectively.

We can write $\vec{\mathbf{x}} = -2\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, so

$$\begin{aligned} A^{25}\vec{\mathbf{x}} &= -2A^{25}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + A^{25}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= -2(2)^{25}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + (-1)^{25}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^{26} \\ -2^{26} \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 2^{26} - 2 \\ -2^{26} - 1 \end{bmatrix} = \begin{bmatrix} 67108862 \\ -67108865 \end{bmatrix} \end{aligned}$$

6. (a) Any basis for R^2, for example, $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

   (b) A^T has 3 columns and $\operatorname{rank}(A^T) = \operatorname{rank}(A) = 2$, so the nullity of A^T is 1.

   (c) No, because the matrix times the vector is not zero.

   (d) For example, the 2×2 identity matrix, or any matrix with two rows and one of them is not a scalar multiple of the other.

   (e) Since the geometric multiplicity of 2 is 4, the dimension of E_2 must be 4. So the nullity of $A - 2I$ must be 4. But the only 4×4 matrix with nullity of 4 is the zero matrix, so $A - 2I = \mathcal{O}$, so

$$A = 2I = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

7. (a) One way to do this is to show that if A did have an inverse, then $(A^{-1})^3A^3 = (A^{-1})^3\mathcal{O}$, so the identity matrix would be equal to the zero matrix, which can't happen. Another way is to argue that $(\det A)^3 = \det(A^3) = 0$, so $\det A = 0$.

   (b) Since A is singular, there is some nonzero $\vec{\mathbf{x}}$ such that $A\vec{\mathbf{x}} = \vec{\mathbf{0}}$. In other words, $A\vec{\mathbf{x}} = 0\vec{\mathbf{x}}$. So 0 is an eigenvalue of A.

   (c) Suppose that $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ for some nonzero $\vec{\mathbf{x}}$. Then $A^3\vec{\mathbf{x}} = \lambda^3\vec{\mathbf{x}}$. But A^3 is zero, so $\lambda^3\vec{\mathbf{x}} = \vec{\mathbf{0}}$. But $\vec{\mathbf{x}}$ is not zero, so $\lambda^3 = 0$. But this means that $\lambda = 0$. So 0 is the only possible eigenvalue.