FINDING RATIONAL ROOTS OF POLYNOMIALS.

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Let $f(t)$ be a polynomial, and suppose that you want to solve the equation $f(t) = 0$. The solutions to this equation are called *roots* or *zeros* of f. You might also want to know how many times each root shows up in the factorization of $f(t)$. This is called the *multiplicity* of a root. For example, if $f(t) = (t-2)(t+10)^4$, then the roots of f are 2 and -10, and the multiplicity of 2 is 1 and the multiplicity of -10 is 4.

There are several algorithms that can be used to find roots, but most of them are too complicated to do without using a computer. I will show you how to find all of the rational roots of a polynomial and their multiplicities. These are the easiest roots to find.

In case you have forgotten, a rational number is a number that can be written in the form of a fraction, like $\frac{1}{2}$, $\frac{2}{3}$, or 4. An irrational number is a number that cannot be written in fraction form, like $\sqrt{2}$ or $\sqrt[3]{5}$.

1. The Rational Roots Theorem

Theorem. Let

$$f(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + \cdots a_1 t + a_0.$$

If r is a root of f, then $r = \pm\frac{b}{c}$, where b is a factor of a_0, and c is a factor of a_n.

This theorem tells us all the possible rational roots of $f(t)$. For example, if $f(t) = 5t^2+29t-6$, then every rational root must be plus or minus some factor of 6 divided by some factor of 5. The factors of 6 are $1, 2, 3$, and 6, and the factors of 5 are 1, and 5, so the possible rational roots are

$$\pm1, \pm2, \pm3, \pm6, \pm\frac{1}{5}, \pm\frac{2}{5}, \pm\frac{3}{5}, \pm\frac{6}{5}.$$

The actual roots are 6 and $\frac{1}{5}$.

Example 1. Find all the rational roots of $f(x) = x^3 - 2x^2 - 5x + 6$.

Solution. The rational roots theorem tells us that the only possibilities are $\pm1, \pm2, \pm3, \pm6$. To find out which ones are really roots, we need to evaluate the polynomial at each value:

$\mathbf{f(1) = 0}$	$f(-1) + 8$	$f(2) = -4$	$\mathbf{f(-2) = 0}$
$\mathbf{f(3) = 0}$	$f(-3) + -24$	$f(6) = 120$	$f(-6) = -252$

So the roots are $1, -2$, and 3. The fundamental theorem of calculus tells us that the number of roots with multiplicity (i.e. the sum of the multiplicities) is the degree of the polynomial. Since the degree of f is 3, we know that these are all the roots of f, and that each root has multiplicity 1. So actually, this tells us $f(x) = (x-1)(x-3)(x+2)$. We factored f using only the rational roots theorem.

Example 2. Find all the rational roots of $g(x) = x^5 + x^4 - x - 1$.

Solution. The only possibilities are ± 1. We check:

$$\mathbf{g(1) = 0} \qquad\qquad \mathbf{g(-1) = 0}$$

So the only rational roots of g are 1 and -1. Since the degree of g is 5, we do not know if these are all the roots of g, and we do not know the multiplicites of the roots.

2. Repeated Root Theorem

Now we want to find out the multiplicity of a root, that is, how many times a certain root is repeated in the polynomial factorization.

Theorem. If r is a root of f and the first $n-1$ derivatives of f are zero at r and the nth derivative is nonzero at r, then the multiplicity of r is n. In symbols, if

$$\begin{aligned} f(r) &= 0 \\ f'(r) &= 0 \\ f''(r) &= 0 \\ &\vdots \\ f^{(n-1)}(r) &= 0 \end{aligned}$$

and

$$f^{(n)}(r) \neq 0$$

then r has multiplicity n.

Example 3. Find the multiplicities of the polynomials in Examples 1 and 2.

Solution. We already know that the roots $f(x)$ from Example 1 have multiplicity 1, but we can also use the repeated roots theorem. Note $f'(x) = 3x^2 - 4x - 5$, and $f'(1) = -6$, $f'(3) = 10$, and $f'(-2) = -1$. None of these are zero, so the multiplicity of each root is 1.

We can do the same thing for g. We see $g'(x) = 5x^4 + 4x^3 - 1$. Also, $g'(1) = 8$, which is not zero, so 1 has multiplicity 1. On the other hand, $g'(-1) = 0$. Now, $g''(x) = 20x^3 + 12x$, and $g''(-1) = 32$, which is not zero. So the multiplicity of -1 is 2, because the 2nd derivative of g is the first one that is nonzero at -1. Now we know all of the rational roots of g and their multiplicies.

Since we have only found 3 roots (1 counted once and -1 counted twice), there must be two more roots out there that are not rational. In other words, we know that $g(x) = (x-1)(x+1)^2 h(x)$, where $h(x)$ is a degree two polynomial. We could use polynomial long division to divide $g(x)$ by $(x-1)(x+1)^2$ and find out $h(x)$, then use the quadratic formula to find the other two roots of $g(x)$. So the rational roots theorem and repeated roots theorem can help us factor a polynomial even if it doesn't have all rational roots.

Example 4 Let $f(x) = x^9 - 3x^8 + 8x^6 - 6x^5 - 6x^4 + 8x^3 - 3x + 1$. All the roots of f are rational. Factor f.

Solutiion. Using the rational roots theorem, the only possible rational roots are 1 and -1. We check that $f(1) = 1$ and $f(-1) = 0$. So these are both roots of f. Now we need to find the multiplicities by taking derivatives:

	Derivative to evaluate	at $x = 1$	at $x = -1$
$f'(x)$	$9x^8 - 24x^7 + 48x^5 - 30x^4 - 24x^3 + 24x^2 - 3$	0	0
$f''(x)$	$72x^7 - 168x^6 + 240x^4 - 120x^3 - 72x^2 + 48x$	0	0
$f'''(x)$	$504x^6 - 1008x^5 + 960x^3 - 360x^2 - 144x + 48$	0	384
$f^{(4)}(x)$	$3024x^5 - 5040x^4 + 2880x^2 - 720x - 144$	0	—
$f^{(5)}(x)$	$15120x^4 - 20160x^3 + 5760x - 720$	0	—
$f^{(6)}(x)$	$60480x^3 - 60480x^2 + 5760$	5760	—

We only have to evaluate each derivative until it is nonzero. At $x = 1$, the 6th deriviative is the first nonzero one, so the multiplicity of 1 is 6. At $x = -1$, the 3rd derivative is the first nonzero on, so the multiplicity of -1 is 3. Since f is degree 9, it has 9 roots, counted with multiplicity, so this is all of the roots. This means $f(x) = (x - 1)^6(x + 1)^3$.

Exercises. Just for practice, find the rational roots and their multiplicities for each of these polynomials. Problems 1—3 have only rational roots.

(1) $x^4 + 8x^3 + 23x^2 + 28x + 12$
(2) $x^5 - 5x^4 - 30x^3 - 50x^2 - 35x - 9$
(3) $x^4 - 3x^3 - 39x^2 + 47x + 210$
(4) $x^5 - 5x^3 + 2x^4 - 10x^2 + 4x + 8$