Null space, range, row space and column space

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Let A be an $m \times n$ matrix (in the illustrations, $m = 2$ and $n = 3$). Multiplication by A is a function whose input is vectors in $\mathbf{R}^n$ and whose output is vectors in $\mathbf{R}^m$. The **null space** of A is the set of all vectors which are sent to zero by this function. The **range** of A is all possible outputs of the function. The **row space** of A is the span of the rows, which is always perpendicular to the null space. The **column space** of A is the span of the columns, which is always the same as the range. This is illustrated in Figure 1.

Things to think about

- What would the picture look like if the null space were 1 dimensional?
- Could the null space possibly be 0 dimensional?
- What will the picture look like if A is a 3×2 matrix and the null space is 0 dimensional? 1 dimensional? 2 dimensional?

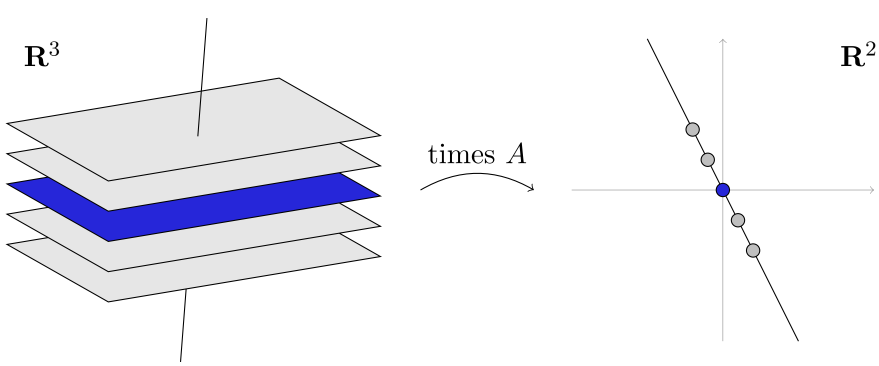
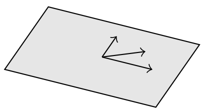
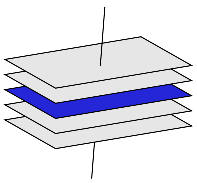
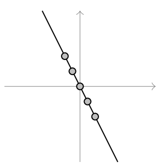
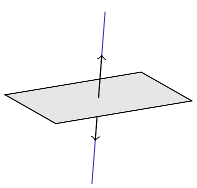
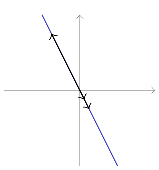


Figure 1: The blue plane on the left is the null space, and A times any point on this plane is zero. The circles on the right represent some outputs you might get when you multiply vectors by A. The line on the right represents the range, which is also the column space. The line on the left is the row space.

Picture	Name	Definition	Finding a basis	Dimension
	Span (of a set)	$\{a_1\mathbf{v}_1+\cdots+a_p\mathbf{v}_p : a_i \in \mathbf{R}\}$	Put the vectors as the rows or columns of a matrix and use the row space or the column space method	
	Null space or kernel (of A)	$\{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$	Find the vector form of the general solution	nullity
	Range or image (of A)	$\{\mathbf{b} : A\mathbf{x} = \mathbf{b}$ is consistent$\}$ or $\{A\mathbf{x} : \mathbf{x} \in \mathbf{R}^n\}$	Use the column space method, or transpose and use the row space method	rank
	Row space (of A)	Span of the rows of A	Use the row space method, or transpose and use the column space method	rank
	Column space (of A)	Span of the columns of A	Use the column space method, or transpose and use the row space method	rank

For these pictures, I used the matrix $A = \begin{bmatrix} 1 & 2 & -6 \\ -2 & -4 & 12 \end{bmatrix}$.

The null space is 2 dimensional. The column space (and range) is 1 dimensional because the three columns are parallel. The row space is 1 dimensional because the two rows are parallel. As is always the case, rank + nullity = number of columns.